SHORT COMMUNICATIONS

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Quasiperiodic tilings with fourfold symmetry.

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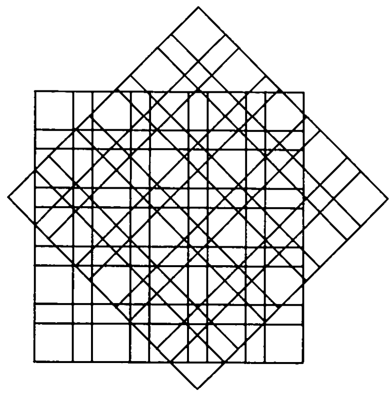
Abstract

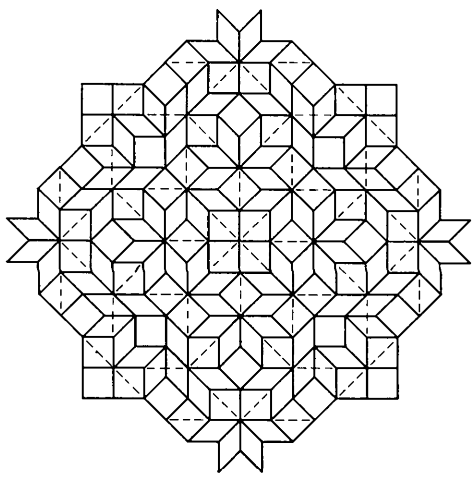
The grid method can be used to construct different quasiperiodic tilings with fourfold symmetry made of rhombuses and squares by altering the topology of the multigrid. Three examples are shown and their differences are highlighted.

Quasicrystals discovered by Shechtman, Blech, Gratias & Cahn (1984) have been very successfully modelled by Penrose tilings (Penrose, 1974; Kramer & Neri, 1984; Elser, 1986; Duneau & Katz, 1985; Gahler & Rhyner, 1986; Bak, 1986). One of the standard methods of constructing the Penrose tilings is the dual-grid method (de Bruijn, 1981). A grid is a set of parallel lines and a multigrid is the superposition of ordinary grids. The dual map to the multigrid constructs the quasiperiodic tiling. In fact, the definition of quasiperiodicity originated from the grid method (Steinhardt & Ostlund, 1987). A discussion of the grid method has been provided by Socolar (1989) for tilings with eight-, ten- and twelvefold symmetries. While the usual discussions on quasiperiodicity are concerned with fivefold symmetry, we earlier showed that nonperiodic tilings with two-, three-, four- and sixfold symmetry can be constructed by self-similarity (Baranidharan, Balagurusamy, Srinivasan, Gopal & Sasisekharan, 1989; Sasisekharan, Baranidharan, Balagurusamy, Srinivasan & Gopal, 1989). Recently, Clark & Suryanarayan (1991) presented a nonperiodic fourfold-symmetry tiling constructed by self-similarity. In the past, there have rarely been published examples of tilings with reduced symmetries that are constructed by the multigrid method although the construction possibility itself is well known. The grid vectors as well as the grid-line spacings can be altered to produce a quasiperiodic tiling with a desired symmetry. Some examples are illustrated here.

The multigrid from which the octagonal tiling is constructed contains a single spacing in each symmetry direction. The symmetry of the octagonal tiling can therefore be reduced by increasing the number of spacings in each direction. Fig. 1(*a*) is a multigrid in which the spacing in each direction is given by *ABABABAB*.... The dual to the grid is constructed with squares and rhombuses and it is shown in Fig. 1(*b*). This tiling possesses perfect fourfold symmetry and yet is quasiperiodic. Since the multigrid in Fig. 1(*a*) can also be considered as an octagonal grid plus a decoration (*i.e.* if we consider *A*, *B* as a single unit then it amounts to a single-spacing grid with a decoration) one can obtain this tiling by projection from a decorated four-dimensional lattice. The octagonal superlattice in the tiling is seen along the dotted lines. The self-similarity of the tiling is evident from the inflated rhombuses and squares. There are two types of inflated rhombuses and one type of inflated square during the inflation. Effectively the symmetry of the octagonal tiling is reduced to four by a decoration while retaining its other properties.



(*a*)



(*b*)

Fig 1. (*a*) The multigrid with two intervals in each direction spaced periodically. (*b*) The dual to (*a*), made of rhombuses and squares. The dotted lines show the octagonal superstructure.

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Another way to construct a quasiperiodic tiling with fourfold symmetry is to use quasiperiodic spacing of the grid lines with two types of spacings. To obtain a multigrid with quasiperiodic spacing we exploit the irrational property of $1+2^{1/2}$. The matrix whose largest eigenvalue is $1+2^{1/2}$ is given by

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

A quasiperiodic sequence of long and short intervals can be obtained by the recursive operation

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} AB \\ ABA \end{pmatrix}.$$

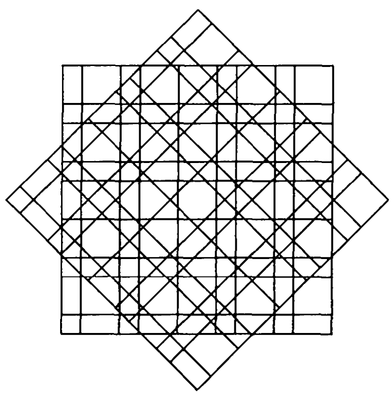
A sequence obtained from the recursion is *ABABAABABAABABABA*.... We could have also written the recursion rule as

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} AB \\ AAB \end{pmatrix}$$

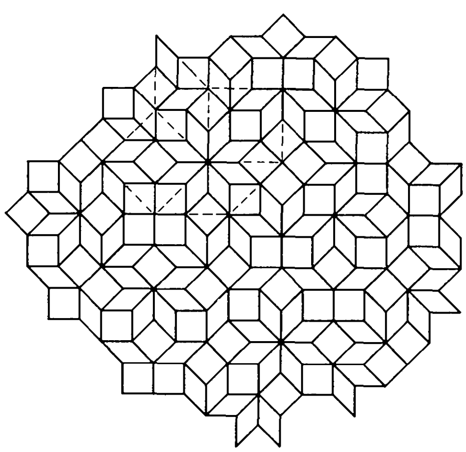
and this will give the same sequence with a shift (although in many other cases like this the same sequence will not necessarily be obtained).

Now the grid can be constructed in which the spacings between the grid lines follow the above sequence. A portion of the multigrid is shown in Fig. 2(*a*). The dual to a section of the multigrid is shown in Fig. 2(*b*). As shown by dotted lines, there are larger and larger squares and rhombuses connecting various vertices characteristic of the self-similarity of the tiling. Self-similar tilings are expected to have strong local rules (Levitov, 1988). This tiling consists of vertices that are not found in the eightfold tiling. This tiling is not locally isomorphic to Fig. 1(*b*). An important aspect of this type of tiling is that, depending on the length scales *A* and *B*, the final tiling will be different. It is not at present clear how to obtain by projection the tilings that are obtained from quasiperiodic grids.

We have seen two examples in which the symmetry is fourfold, self-similarity is present and yet not locally isomorphic. There is also another example where the symmetry is fourfold and there is no self-similarity. To construct this tiling we start from a periodic but inequivalent grid. Fig. 3(*a*) shows grids with a single spacing in each direction but the spacings are different for the two grids. Fig. 3(*b*)

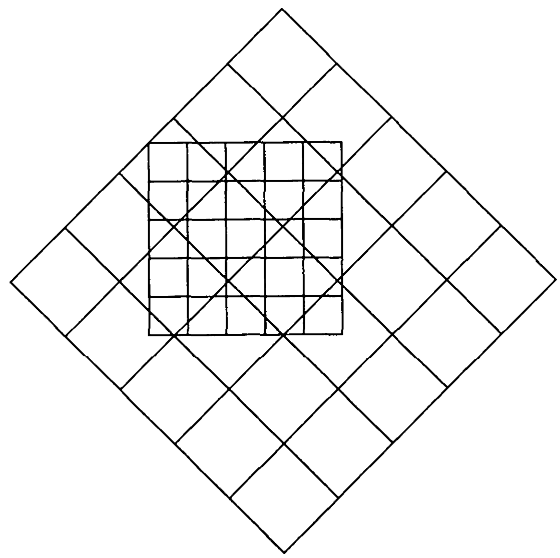


(*a*)

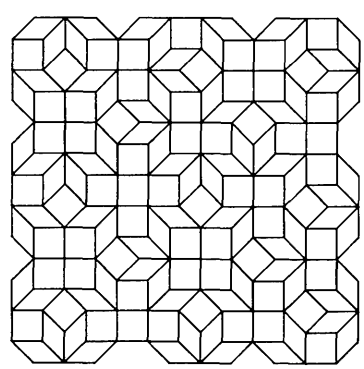


(*b*)

Fig. 2. (*a*) The multigrid in which the grid spacings are quasiperiodic. (*b*) A quasiperiodic tiling with fourfold symmetry constructed from (*a*).



(*a*)



(*b*)

Fig. 3. (*a*) The multigrid with a single periodic but inequivalent spacing. (*b*) The dual to (*a*) is a quasiperiodic tiling that lacks self-similarity.

is a tiling corresponding to Fig. 3(*a*). Alternatively, this tiling may be obtained by the projection method in which the four-dimensional hyperlattice spanned by four orthonormal basis vectors (**e**1, 2**e**2, **e**3, 2**e**4) is divided into two subspaces (**e**1, **e**3 and 2**e**2, 2**e**4) each of two dimensions. Fig. 3(*b*) is not locally isomorphic to Fig. 1(*b*) or Fig. 2(*b*). Also, Fig. 3(*b*) does not possess the self-similarity property. It is therefore reinforced that not all quasiperiodic tilings have self-similarity as an essential property and therefore the self-similarity property cannot be used to establish the quasiperiodicity of a tiling, in contrast to the view of Clark & Suryanarayan (1991). Since quasiperiodicity implies a continuum in a higher-dimensional space, one should 'lift' (Levitov, 1988) a tiling obtained by self-similarity to a higher space to prove its quasiperiodicity.

One can construct quasiperiodic tilings with rotational symmetries of order two, three and six by suitable choice of the starting grid with more than one spacing. The construction of three-dimensional tilings from a decorated four-dimensional lattice leads to potential models for incommensurate phases when the projection is made onto the octahedron.

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Books Received

Acta Cryst. (1992). A**48**, 784

The following books have been received by the Editor. Brief and generally uncritical notices are given of works of marginal crystallographic interest; occasionally a book of fundamental interest is included under this heading because of difficulty in finding a suitable reviewer without great delay.

High-resolution transmission electron microscopy. Edited by PETER BUSECK, JOHN COWLEY and LEROY EYRING. Pp. xxii+645. Oxford University Press, Oxford and New York, 1988. Price £50.00. ISBN 0-19-504275-1. A review of this book, by Alain Bourret, has been published in the June 1992 issue of *Journal of Applied Crystallography*, pages 463–464.